ESSENTIAL AND INESSENTIAL POINTS OF LOCAL NONCONVEXITY

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ABSTRACT

Let S be a closed connected subset of a Hausdorff linear topological space, Q the points of local nonconvexity of S, E the essential members of Q, N the inessential. If $S \sim Q$ is connected, then the following are true: Theorem 1. *If* $Q \neq \emptyset$ *is countable, then* S *is planar.* Theorem 2. *If* Q *is finite and nonempty, then* card $E \geqq$ card $N + 1$. Theorem 3. *If* $S \subseteq R^2$ *and* N *is infinite, then* E *is infinite.*

1. Introduction

Let S be a subset of a Hausdorff linear topological space. A point x in S is a *point of local convexity of* S (alternately, S is locally convex at x) iff there is some neighborhood U of x such that if $y, z \in S \cap U$, then $[y, z] \subseteq S$. If S fails to be locally convex at some point q in S, then q is called a *point of local nonconvexity* (lnc point) *of* S.

Guay and Kay [1] have introduced the concepts of essential and inessential points of local nonconvexity, defined below.

DEFINITION. A point q in S is called an *essential point of local nonconvexity of* S iff for every neighborhood U of q there is at least one component W of $S \cap U \sim \{q\}$ such that q is an lnc point of cl W; an lnc point of S which is not essential is called an *inessential point of local nonconvexity of* S.

Throughout the paper, Q denotes the set of lnc points of S; E and N denote the sets of essential and inessential points of Q, respectively. S is closed and connected, and $S \sim Q$ is connected.

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2. A generalization of the theorem of Guay and Kay

In [1] it is proved that if Q is finite and nonempty, then S is planar. Theorem 1 of this paper is a direct generalization to the case for Q countable. The following lemmas will be used in the proof.

LEMMA 1. *Let $S \subseteq R^k$, Q countable, and let n denote the largest integer j for which the following is true: There is a convex subset C of S with* $\dim \operatorname{aff} C = j$. *If $s \in S \sim Q \neq \emptyset$, then for every neighborhood U of s with $U \cap S$ convex,* $\dim \operatorname{aff}(U \cap S) = n$. *That is, U contains interior points of S in some n-dimensional flat.*

PROOF. Assume the lemma fails for some neighborhood U of s, $U \cap S$ convex. Then $\dim \operatorname{aff}(U \cap S) < n$ and, by our hypothesis, there are points of S not in $\operatorname{aff}(U \cap S)$. Furthermore, since Q is countable and $S \sim Q$ is connected, we may select some point z of $S \sim Q$, $z \notin \operatorname{aff}(U \cap S)$.

Now, since $S \sim Q$ is connected and locally convex, it is polygonally connected [2], and there is a polygonal path λ in $S \sim Q$ from s to z. However, $\operatorname{aff}(U \cap S) \cap \operatorname{cl}(S \sim \operatorname{aff}(U \cap S)) \subseteq Q$, so some point of λ must lie in Q, a contradiction. Therefore, the lemma must hold.

LEMMA 2. *If $S \subseteq R^k$, $S \sim Q \neq \emptyset$, and Q is countable, then* $\operatorname{int} S \neq \emptyset$ *(as a subset of* $\operatorname{aff} S$).

PROOF. We assume $\operatorname{int} S = \emptyset$ in order to reach a contradiction. Since Q is countable, $S \sim Q$ necessarily has nontrivial convex subsets. Let n denote the largest integer j for which a convex subset of $S \sim Q$ has interior points in R^j, and let C be a convex subset of $S \sim Q$ for which $\dim \operatorname{aff} C = n$.

By our assumption, there are points of S not in $\operatorname{aff} C$. Moreover, since Q is countable and S is connected, there are points of $S \sim Q$ not in $\operatorname{aff} C$. Using the fact that $(\operatorname{aff} C) \cap \operatorname{cl}(S \sim \operatorname{aff} C) \subseteq Q$, an argument similar to the proof of Lemma 1 shows that $S \sim Q$ cannot be polygonally connected. This is a contradiction. Hence our assumption is false and we conclude that $\operatorname{int} S \neq \emptyset$.

COROLLARY. *If $S \subseteq R^k$, Q is countable, and $s \in S \sim Q \neq \emptyset$, then every neighborhood of s contains points of* $\operatorname{int} S \neq \emptyset$.

The following result by Gauy and Kay [1] is also needed.

LEMMA 3. *If $[x, y] \cup [y, z] \subseteq S$ and no point of Q lies in* $\operatorname{conv}\{x, y, z\} \sim [x, z]$, *then* $\operatorname{conv}\{x, y, z\} \subseteq S$.

THEOREM 1. *If $Q \neq \emptyset$ is countable, then S is planar.*

PROOF. Let x, y be points of $S \sim Q$ such that $[x, y] \nsubseteq S$. Since $S \sim Q$ is polygonally connected, we may assume, without loss of generality, that for some z in $S \sim Q$, $[x, z] \cup [z, y] \subseteq S \sim Q$. Let π denote the plane determined by x, y, z.

If $S \nsubseteq \pi$, there exists some point p of S not in π. Since Q is countable and S is connected, p may be chosen in $S \sim Q$. Moreover, since $S \sim Q$ is polygonally connected, we may select p so that $[y, p] \subseteq S \sim Q$.

Let $Y \equiv \text{aff}\ (\pi \cup \{p\})$ and let T be the connected component of $(S \sim Q) \cap Y$ containing $\{x, y, z, p\}$. Then $S_0 \equiv \text{cl}\ T$ is a closed, connected subset of Euclidean 3-space and every lnc point of S_0 is in Q. Also, since $T = T \sim Q \subseteq \text{cl}\ T \sim Q \subseteq \text{cl}\ T$, and T is connected, $\text{cl}\ T \sim Q = S_0 \sim Q$ is connected. For the remainder of the proof we will examine S_0 as a subset of R^3.

By the corollary to Lemma 2, int $S_0 \neq \emptyset$ and every neighborhood U of y (in R^3) contains points of int S_0. Thus U contains points of S_0 not in π. Assume that U is chosen so that $U \cap S_0$ is convex. For some u in $(U \cap \text{int}\ S_0) \sim \pi$, conv $\{z, y, u\} \sim [u, z]$ contains no point of Q, for otherwise, Q could not be countable. Since $[z, y] \cup [y, u] \subseteq S_0 \sim Q$, by Lemma 3, conv $\{z, y, u\} \subseteq S_0$. Without loss of generality, we may assume that u is selected so that conv $\{z, y, u\} \subseteq S_0 \sim Q$. Since $S_0 \sim Q$ is open in S_0, there is a neighborhood of conv $\{z, y, u\}$ containing no member of Q. There exists a neighborhood V of u so that for all t in $V \cap S$, $[z, t] \subseteq S_0 \sim Q$.

Recall that V contains points of int S_0. For some t_0 in V, conv $\{x, z, t_0\} \sim [x, t_0]$ contains no point of Q. Since $[x, z] \cup [z, t_0] \subseteq S_0 \sim Q$, again, by Lemma 3, conv $\{x, z, t_0\} \subseteq S_0$, and we may select t_0 so that conv $\{x, z, t_0\} \subseteq S_0 \sim Q$.

There is a neighborhood W of t_0 so that $[x, t] \subseteq S_0 \sim Q$ for all t in $W \cap S_0$.

We have $[x, t] \cup [t, y] \subseteq S_0 \sim Q$ for all t in $W \cap U \cap S_0$. For some t_1 in $W \cap U \cap S_0$, conv $\{x, t_1, y\} \sim [x, y]$ contains no point of Q. By Lemma 3, this implies that $[x, y] \subseteq S_0$, so $[x, y] \subseteq S$, a contradiction. Therefore $S \subseteq \pi$, completing the proof.

3. The cardinalities of E and N

We are interested in the relationship between card E and card N for both finite and infinite Q. Guay and Kay have conjectured that for Q finite, card $E \geqq \frac{1}{2}$ (card $Q + 1$), and Theorem 2 shows that this is correct.

THEOREM 2. *Let Q be finite and nonempty; and let* $m \equiv \text{card}\ N$, $e \equiv \text{card}\ E$.

Then $e \geqq m+1$. *Moreover, the bound of* $m+1$ *is best possible for every* m.

PROOF. Clearly S is planar. The proof is by induction. If $m \geqq 1$, let q be an inessential lnc point of S. There is some neighborhood U of q such that for each component W of $S \cap U \sim \{q\}$, q is not an lnc point for cl W. Clearly there are at least two components W_1, W_2 of $S \cap U \sim \{q\}$, and without loss of generality, we assume there are exactly two. For U sufficiently small, $U \cap Q = \{q\}$, $W_i \cup \{q\}$ is convex and is contained in some maximal convex subset C_i of S, $i = 1,2$. Moreover, the connectedness of $S \sim Q$ implies that each C_i contains some essential lnc point p_i of S. It remains to show that $p_1 \neq p_2$.

Let x belong to $C_1 \cap C_2$. If $x \neq q$, then $[x,q] \subseteq C_1 \cap C_2$ and $[x,q) \cap U$ necessarily lies in $W_1 \cap W_2$, a contradiction, since W_1, W_2 are disjoint. Thus $C_1 \cap C_2 = \{q\}$, $p_1 \neq p_2$, and if $m \geqq 1$, $e \geqq 2$.

Assume that for $m < k$, if S has at least m inessential lnc points, then S has at least $m+1$ essential lnc points. Suppose S has k inessential lnc points.

Examine the connected components of $X \sim S$. Since $k \geqq 1$, $X \sim S$ has bounded components, and since Q is finite, there are finitely many components.

Let $\{V_0, V_1, \cdots\}$ be a maximal sequence of bounded components of $X \sim S$ such that V_i, V_{i+1} are adjacent (i.e. bdry $V_i \cap$ bdry $V_{i+1} \neq \emptyset$), and for $q_i \in$ bdry $V_i \cap$ bdry V_{i+1}, $q_i \notin$ bdry V_{i+2}, $0 \leqq i$. Clearly for $q_i \in$ bdry $V_i \cap$ bdry V_{i+1}, $q_i \in Q$. Since $S \sim Q$ is connected, the V_i sets are necessarily distinct and the sequence is finite. Similarly, at most one V_i is adjacent to an unbounded component of $X \sim S$, so, without loss of generality, we may assume that V_0 is not adjacent to an unbounded component. Clearly cl (V_0) contains at least three lnc points of S, and at least two of these are distinct from q_0. Let $Q_0 \equiv$ cl $(V_0) \cap Q$.

For $c \in Q_0 \sim \{q_0\}$, there can be only one component of $X \sim S$ containing c in its closure, since our sequence is maximal. Thus V_0 is the only component containing c, c must be an essential lnc point for S and not even an lnc point for cl $(S \cup V_0) \equiv S_0$.

Certainly S_0 is closed, $S_0 \sim Q$ is connected, and S_0 has either k or $k-1$ inessential lnc points. By the induction hypothesis, S_0 has $\geqq k$ essential lnc points.

If q_0 is essential for S_0, then S_0 contains at least $k-1$ essential lnc points which are essential for S. Also $Q_0 \sim \{q_0\}$ contains at least 2 essential lnc points for S which are not lnc points for S_0, so S has at least $(k-1)+2 = k+1$ essential lnc points.

If q_0 is not essential for S_0, then every essential lnc point for S_0 is essential for S. Since $Q_0 \sim \{q_0\}$ contains at least two essential lnc points for S which we have not counted, S has at least $k+2$ essential lnc points, completing the proof.

Moreover, the bound of $m+1$ is best possible, as the following example shows.

EXAMPLE 1. Let S be the set in Fig. 1. For any k, an appropriate adaptation of the figure yields $m = k$, $e = k+1$.

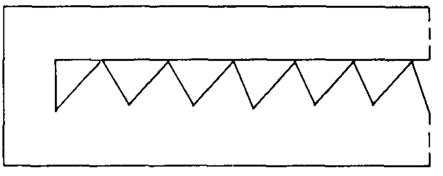

Fig. 1.

Since Theorem 2 yields such a pleasant relationship between the cardinalities of E and N, one might expect a similar result to hold in case N is infinite. The problem is more difficult (and more suprising) than it appears, as Theorem 3 and Example 2 demonstrate.

DEFINITION. Let T be a closed subset of R^k. We say bdry T is *locally arc connected* at $x \in$ bdry T iff there is a sphere U about x such that for V a sphere about x and $V \subseteq U$, $V \cap$ bdry T is arc connected.

THEOREM 3. *If S is planar and N is infinite, then E is infinite.*

PROOF. We begin with a definition. Let $\mathscr{B} = \{B; B$ is the closure of a bounded component of $X \sim S\}$. We call $\mathscr{V} = \{V_0, V_1, \cdots\}$ a chain in cl $(X \sim S)$ iff the following are true:

1) $V_i \in \mathscr{B}$, $0 \leqq i < \infty$
2) for $V_i \in \mathscr{V}$, $0 < i$, there is some V_k in $\mathscr{V}$, $k < i$, such that $V_i \cap V_k \neq \varnothing$.

By an obvious application of Zorn's lemma, any chain is contained in a maximal chain. If $\mathscr{V}$, $\mathscr{W}$ are maximal chains, then $\mathscr{V} \cap \mathscr{W} = \varnothing$.

To prove the theorem, consider two cases.

Case I. Assume that for every lnc point x, bdry S is locally arc connected at x. We will show that every inessential lnc point of S is on the boundary of a member of $\mathscr{B}$. For q an inessential lnc point, choose a neighborhood U of q such that q is not an lnc point for cl K for any component K of $U \cap S \sim \{q\}$. Clearly U may be selected so that $q \in \text{cl } K$ for each component K, for otherwise we would violate local arc connectedness. Therefore, $(X \sim S) \cap U$ has at least two components, and we will show that at least one of these components must lie in a bounded component of $X \sim S$, and q is indeed on the boundary of a member of $\mathscr{B}$.

We begin by proving the following remark.

REMARK 1. *Each component K_0 of $U \cap S \sim \{q\}$ contains points in $S \sim Q$.*

PROOF. Since q is not an lnc point for cl K_0, there is some neighborhood U_0 of q, $U_0 \subseteq U$, for which $U_0 \cap \text{cl } K_0$ is convex. For x in $U_0 \cap K_0$, we assert that there is a neighborhood of x disjoint from every other component of $U \cap S \sim \{q\}$: Suppose that for every neighborhood N of x, $N \subseteq U_0 \sim \{q\}$, N contained points in other components of $U \cap S \sim \{q\}$. Then since the sets $K \cap N$, K a component of $U \cap S \sim \{q\}$, are pairwise disjoint, $N \cap \text{bdry } S$ could not be arc connected, violating the definition of local arc connectedness. Thus there is a neighborhood N_0 of x, $N_0 \subseteq U_0 \sim \{q\}$, for which $N_0 \cap S \subseteq K_0$. Since cl K_0 is convex, x is a point of local convexity of S, and $x \in S \sim Q$, completing the proof of the remark.

Now if more than one component of $(X \sim S) \cap U$ lay in an unbounded component of $X \sim S$, then using Remark 1, we could select appropriate points of $(S \sim Q) \cap U$ which could not be joined by a polygonal path in $S \sim Q$, a contradiction. Thus some component of $(X \sim S) \cap U$ lies in a bounded component of $X \sim S$, and q is on the boundary of a member of $\mathscr{B}$.

Case I 1. For the present, we assume that each member of $\mathscr{B}$ contains at most a finite number of inessential lnc points of S.

Case I 1a. Suppose that for every maximal chain $\mathscr{M}$, $\cup \mathscr{M}$ contains at most a finite number m of inessential lnc points. If any chain contains infinitely many essential lnc points, the proof is complete. Otherwise, each chain is finite. Thus $S_m \equiv S \cup \{B; B \in \mathscr{B}, B \notin \mathscr{M}\}$ has m inessential lnc points and at least $m+1$ essential lnc points, using Theorem 2. For each maximal chain $\mathscr{M}$, since $\mathscr{M}$ consists of finitely many members of $\mathscr{B}$, every essential lnc point for S_m is necessarily essential for S, and the lnc points corresponding to each S_m are unique. Since N is infinite, there are infinitely many chains and infinitely many essential lnc points of S.

Case I 1b. Suppose for some maximal chain $\mathscr{M}$, $\cup \mathscr{M}$ contains infinitely many inessential lnc points. We assume that S has a finite number e of essential lnc points to reach a contradiction.

Since every member of $\mathscr{B}$ contains at most finitely many inessential lnc points of S, clearly we may select some finite subchain $\mathscr{V}$ of $\mathscr{M}$, $\cup\ \mathscr{V}$ containing at least e inessential lnc points of S. For each inessential lnc point q in $\cup\ \mathscr{V}$, select V_{q_1}, V_{q_2} in $\mathscr{M}$, $q \in V_{q_1} \cap V_{q_2}$. Let $\mathscr{W} \equiv \mathscr{V} \cup \{V_{q_i}; i = 1,2\}$. Then $S_w \equiv S \cup \{V; V \notin \mathscr{W}\}$ has at least e inessential lnc points and at most a finite number (since $\mathscr{W}$ is finite).

Therefore by Theorem 2, S_w has at least $e + 1$ essential lnc points. However, these are not necessarily essential for S.

For x essential for S_w and not for S, there is a maximal subchain of $\mathscr{M}$, call it $\mathscr{U}_x \equiv \{U_0, U_1, \cdots\}$ with $x \in U_0$ and $U_0 \notin \mathscr{W}$. Moreover, no U_i belongs to $\mathscr{W}$. For otherwise $(\cup\, \mathscr{U}_x) \cup (\cup \mathscr{W})$ would contain a closed curve λ in $(X \sim S) \cup Q$ with $x \in \lambda$. Since x is not essential for S, x is inessential for S, and by the argument in Remark 1, distinct points of $S \sim Q$ would necessarily lie interior to λ and exterior to λ, contradicting the polygonal connectedness of $S \sim Q$.

Similarly, if x, y are distinct essential lnc points for S_w and not for S, $\mathrm{cl}\,(\cup \mathscr{U}_x) \cap \mathrm{cl}\,(\cup \mathscr{U}_y) = \emptyset$. Again using connectedness and our hypothesis for case I 1, $\cup \mathscr{U}_x$ contains some essential lnc point of S.

Thus for each x essential for S_w, there corresponds a unique x_0 essential for S and S has at least $e + 1$ essential lnc points. This contradiction implies that E is infinite, completing case I 1.

Case I 2. Suppose for some B in $\mathscr{B}$, bdry B contains infinitely many inessential lnc points. Clearly no two inessential lnc points in B lie in the same member of $\mathscr{B} \sim \{B\}$, for by an argument similar to that used in case I 1b above, this would violate the polygonal connectedness of $S \sim Q$. Also, using Remark 1 and the polygonal connectedness of $S \sim Q$, at most one inessential lnc point q_0 in B can lie on the boundary of an unbounded component of $X \sim S$. Thus to each $q \neq q_0$ in B, there corresponds a unique member B_q of $\mathscr{B} \sim \{B\}$. Again, by connectedness of $S \sim Q$, the B_q sets are pairwise disjoint. For each B_q, let C_q denote the closure of a component of $[(X \sim S) \sim B] \cup Q$ containing B_q. The C_q sets are disjoint, and at most one C_q can be unbounded. To each bounded C_q, there corresponds a unique essential lnc point of S, and case I is finished.

Case II. Assume that for some lnc point q of S, bdry S is not locally arc connected at q. Since $X \sim S$ is open, for each positive integer n and each $1/n$ sphere D_n about q, there is some ray from q containing x_n, y_n in $S \cap D_n$ for which $(x_n, y_n) \subseteq X \sim S$.

Case II1. For the moment, assume that we may choose x_n, y_n distinct from q. For each n, let C_{n1}, C_{n2} denote the closures of distinct components of $[(X \sim S) \sim (x_n, y_n)] \cup Q$ with $(x_n, y_n) \subseteq C_{ni}$, $i = 1,2$. We may select an infinite collection $\mathscr{C}$ of bounded disjoint C_{ni} sets, for otherwise it would violate the connectedness of $S \sim Q$. To each C_{ni} in $\mathscr{C}$, there corresponds a unique essential lnc point. Thus E is infinite.

Case II2. Suppose that there is some integer k such that for all $n \geqq k$, one of x_n, y_n must be q. Select y_0 in $S \cap D_k$ with $(q, y_0) \subseteq X \sim S$. Let U_0 be the open sphere having center q, radius $|q - y_0|$. Since $X \sim S$ is open, there is an open sector of the disk U_0 lying in $X \sim S$ and containing (q, y_0). Let V_0 denote the maximal open sector of U_0 having these properties. (Without loss of generality, we may assume that the arc for V_0 has length less than π, for this fails to occur at most once in our inductive construction.) If the segments $[q, a]$, $[q, b]$ and the arc ab bound V_0, then each of $(q, a]$, $(q, b]$ contains some point of S. Thus we may select points s_0, t_0 in $S \cap \text{bdry } V_0$ with (s_0, t_0) interior to V_0. Then $(s_0, t_0) \subseteq X \sim S$. Let C_0 be the closure of a component of $[(X \sim S) \sim (s_0, t_0)] \cup Q$ with $(s_0, t_0) \subseteq C_0$.

Inductively, proceed as follows: Assume V_j, C_j are defined for $0 \leqq j < i$. Since bdry S is not locally arc connected at q, there is some point y_i of $S \cap D_{k+i}$ not on bdry V_j, $0 \leqq j < i$, and with $(q, y_i) \subseteq X \sim S$. Define V_i, s_i, t_i as previously shown. If C_i is the closure of a component of $[(X \sim S) \sim (s_i, t_i)] \cup Q$ with $(s_i, t_i) \subseteq C_i$, C_i may be selected so that it is disjoint from each C_j, $0 \leqq j < i$. Since each bounded C_i contains an essential lnc point of S and at most one C_i is unbounded, there are infinitely many essential lnc points of S, completing case II and finishing the proof.

COROLLARY. *If Q is countable, then* card $E \geqq$ card N.

It is interesting to note that N may be uncountable while E is countable, as Example 2 shows.

EXAMPLE 2. Let S be the set in Fig. 2. Let x be the point $(0,1)$, y the origin. Then every point of the segment $[x, y)$ is an inessential lnc point of S, while $E = \{(1/n, 0)\} \cup \{y\}$.

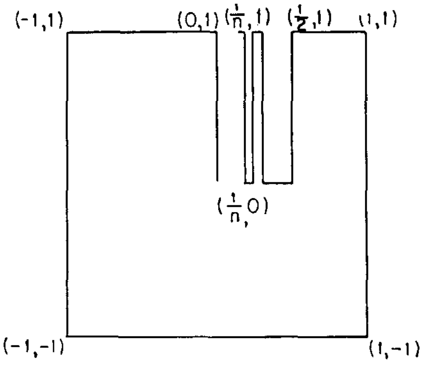

Fig. 2.

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